ON CERTAIN HOMOMORPHISMS OF RESTRICTION ALGEBRAS OF SYMMETRIC SETS(1)

BY

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ABSTRACT

Let G be a locally compact abelian group and Γ its dual group. For any closed $H \subseteq G$ denote the algebra of restrictions to H of Fourier transforms of functions in $L^1(\Gamma)$ by $A(H)$. This paper considers certain Cantor like sets in R and $\Pi\ Z_{m(j)}$ and gives some necessary algebraic criterion for *natural* isomorphisms of their restriction algebras.

0. **Introduction**. Let R be the real line and $A = A(R)$ the space of continuous function on R which are the Fourier transforms of functions in $L^1(R)$. $A(R)$ is a Banach algebra when it is given the $L^1(R)$ norm. Beurling and Helson [2] established that every automorphism of A arises from a map ϕ by $f \to f \circ \phi$ where $\phi(x) = ax + b$. For a closed $F \subseteq R$ one defines $A(F)$ as the restrictions of $f \in A$ to F with the norm of $g \in A(F)$ the infimum of the norms of elements of A whose restrictions are g. One may ask for existence and characterizations of isomorphisms between Banach algebras $A(F_1)$ and $A(F_2)$. In [5] it is shown that such an isomorphism of norm one must be given by $f \to f \circ \phi$ where ϕ: $F_2 \to F_1$ is continuous and for some complex number c of modulus one $ce^{i\phi}$ is a restriction to F_2 of a character of the discrete reals. Further, if F_2 is thick in some appropriate sense the character is continuous.

In this paper we shall consider isomorphisms of restriction algebras of sets of the form

$$F_\gamma = \{ \sum \varepsilon_j \gamma^j : \varepsilon_j \text{ either } 0 \text{ or } 1\}$$

where $\gamma < 1/2$. We shall obtain necessary algebraic conditions on γ_1 and γ_2 for $A(F_{\gamma_1})$ to be isomorphic to $A(F_{\gamma_2})$ induced by $\phi: \sum\varepsilon_j\gamma_1^j \to \sum\varepsilon_j\gamma_2^j$. In particular if γ_1 and γ_2 are algebraic they must be conjugates. We shall also consider sets $E_m \subseteq \prod_1^\infty Z_{m(j)}$ of the form

$$E_m = \{x : j\text{th coordinate is } 0 \text{ or } 1\}.$$

We shall obtain necessary conditions for isomorphisms of $A(E_m)$ with $A(F_\gamma)$ to be induced by $\phi: x \to \sum x_j\gamma^j$, where x_j is the jth coordinate of x. We shall also study necessary conditions for ϕ^{-1} to induce isomorphisms.

1. **Definitions and notations.** For background material and notation not defined here we refer the reader to [7] and [15].

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In this paper G will always be a locally compact abelian group with dual group Γ. If g and γ are elements of G and Γ respectively, the value of the character γ at the point g will be denoted by (γ, g).

When we have a sequence of compact abelian groups G_j, we shall denote their *direct product* (complete direct sum [15]) by $\prod G_j$. If Γ_j is the dual of G_j, then the direct sum [15] $\sum \Gamma_j$ is the dual of $\prod G_j$. The jth *coordinate* of elements g of $\prod G_j$ or γ of $\sum \Gamma_j$ will be denoted by g_j and γ_j. One has:

$$(\gamma, g) = \prod (\gamma_j, g_j)$$

where all but a finite number of elements in the product are 1.

We shall be dealing with the following basic groups:

(i) R will denote the *additive group* of *reals*. R is isomorphic to its dual under the pairing given by

$$(y, x) = \exp(xy),$$

with $x, y \in R$ and $\exp(xy) = e^{2\pi ixy}$.

(ii) Z_n for $n \geqq 2$ will denote the *additive group* of *integers mod n*. Z_n is also isomorphic to its dual under the pairing given by

$$(r, s) = \exp(rs/n),$$

$r, s \in Z_n$.

We shall adopt the notation and assume familiarity with the results in [15] about $L^1(G)$, $M(G)$, the Fourier transforms $\hat{f}$ and $\hat{\mu}$ of $f \in L^1(G)$ and $\mu \in M(G)$, and the convolutions $f * g$ and $\mu * \nu$ of functions in $L^1(G)$ and measures in $M(G)$.

Let $A = A(G)$ be defined by

$$A(G) = \{\hat{f}: f \in L^1(\Gamma)\}.$$

$A(G)$ is a Banach algebra under pointwise multiplication and with norm $\| \cdot \|_A$ defined by $\|\hat{f}\|_A = \|f\|_{L_1(\Gamma)}$ and is isomorphic to $L^1(\Gamma)$ under $*$. For a closed set $E \subseteq G$ define the restriction algebra

$$A(E) = \{\hat{f}/E: f \in L^1(\Gamma)\}$$

with norm $\| \cdot \|_{A(E)}$ defined by

$$\| h \|_{A(E)} = \inf\{ \|\hat{f}\|_A : \hat{f}/E = h\}.$$

$A(E)$ is again a Banach algebra under pointwise multiplication. Set

$$I(E) = \{\hat{f}: \hat{f}/E = 0 \text{ and } f \in L^1(\Gamma)\}$$

$A(E)$ can be identified with the quotient algebra $A(G)/I(E)$.

The dual space of $A(G)$ is denoted by PM (or $PM(G)$). Its elements are called pseudomeasures. Each $S \in PM$ can be identified with a function $\hat{S} \in L^\infty(\Gamma)$ as follows. The action of $S \in PM$ as a linear functional on $\hat{f} \in A(G)$ is given by

$$(S,\hat{f}) = \int_{\gamma} f(\gamma)\overline{\hat{S}(\gamma)}\,d\mu_{\Gamma}.$$

We shall denote by $\|S\|_{PM}$ the $L^{\infty}(\Gamma)$ norm of $\hat{S}$. Thus PM under $\|\cdot\|_{PM}$ is identical with $L^{\infty}(\Gamma)$ under the sup norm.

Since $A(E)$ is the quotient of $A(G)$ by $I(E)$, the dual of $A(E)$ consists of those $S \in PM$ which annihililate every function in $I(E)$. We shall denote this dual of $A(E)$ by $N(E)$. The set of all $\mu \in M(G)$ with support in E we denote by $M(E)$. $M(E)$ can be considered a subspace of $N(E)$ with $(\mu,\hat{f}) = \int \hat{f}\,d\bar{\mu}$. The two definitions for $\hat{\mu}$ coincide.

If G_1 and G_2 are locally compact abelian groups and E_1 and E_2 are closed subsets of G_1 and G_2 respectively we say that Φ: $A(E_1) \to A(E_2)$ is an isomorphism into iff it is an injective algebraic homomorphism and is continuous. If the range of Φ is dense in $A(E_2)$ there exists a homeomorphism $\phi: E_2 \to E_1$ with $\Phi f = f \circ \phi$ [9]. We always denote the adjoint of Φ taking $N(E_2)$ into $N(E_1)$ by Φ^*.

Symmetric sets in R are defined as follows. For any sequence $r = \{r(j): j = 1,\cdots\}$ of positive real numbers with the property

$$\sum_{k}^{\infty} r(j) < r(k-1)$$

we define the subset F_r of R by

$$F_r = \left\{ \sum_{1}^{\infty} \varepsilon_j r(j) : \varepsilon_j \text{ either 0 or 1} \right\}.$$

The representation of the elements of F_r as an infinite sum is unique. For each positive integer k, the subset F_r^k of F_r is defined by

$$F_r^k = \left\{ \sum_{1}^{k} \varepsilon_j r(j) : \varepsilon_j \text{ either 0 or 1} \right\}.$$

We define $F(l,n)$ by

$$F(l,n) = \left\{ x : x = \sum_{l}^{n} \varepsilon_j r(j) \right\}.$$

We define the subspace $N_1(F_r)$ of $N(F_r)$ by

$$N_1(F_r) = \bigcup_{k=1}^{\infty} M(F_r^k).$$

For $\nu \in N_1(F_r)$ we note

$$\|\nu\|_{PM} = \sup_{x} \left| \sum_{\varepsilon_j} \nu(\{\sum \varepsilon_j r(j)\}) \exp(x \cdot \sum \varepsilon_j r(j)) \right|.$$

For any given sequence $m = \{m(j): j = 1,2,\cdots\}$ of positive integers we define the subset E_m of $\prod_j Z_{m(j)}$ by

$$E_m = \{x\colon x \in \prod Z_{m(j)};\ x_j \text{ either } 0 \text{ or } 1\}.$$

For each positive integer k the subset E_m^k of E_m is defined by

$$E_m^k = \{x\colon x \in E_m;\ x_j = 0 \text{ if } j > k\}.$$

We define $E(l,n)$ by

$$E(l,n) = \{x\colon x \in E \text{ and } x_j = 0 \text{ if } j < l \text{ or } j > n\}.$$

Define the subspace $N_1(E_m)$ of $N(E_m)$ by

$$N_1(E_m) = \bigcup_{k=1}^{\infty} M(E_m^k).$$

For $\nu \in N_1(E_m)$ we note

$$\| \nu \|_{PM} = \sup_{\zeta_j} \left| \sum_x \nu(\{x\}) \zeta_1^{x_1} \cdots \zeta_k^{x_k} \right|$$

where ζ_j are $m(j)$th roots of unity.

The following maps will be called standard homeomorphisms:

(i) $\phi\colon E_m \to F_r$ takes $x \to \sum x_j r(j)$. Denote the inverse of ϕ by ψ.

(ii) $\phi\colon F_r \to F_s$ takes $\sum \varepsilon_j r(j) \to \sum \varepsilon_j s(j)$.

Let θ be a function that represents any of the standard homeomorphisms. We note that θ induces maps between $C(E_m)$ or $C(F_s)$ and $C(F_r)$ by

$$\Theta(f) = f \circ \theta.$$

The adjoint Θ^* is a map between $M(F_r)$ and $M(E_n)$ or $M(F_s)$ and in particular is a map between $N_1(F_r)$ and $N_1(E_m)$ or $N_1(F_s)$ given by

$$\Theta^*(\nu)(\{\theta(x)\}) = \nu(\{x\}).$$

We shall frequently write E for E_m, E^k for E_m^k, F for F_r and F^k for F_r^k.

2. **Necessary conditions for isomorphisms.** Let H_1 and H_2 represent any of the sets E_m or F_r and $\theta\colon H_1 \to H_2$ the standard homeomorphism. If $\Theta\colon C(H_2) \to C(H_1)$ has $\Theta(A(H_2)) \subseteq A(H_1)$ then Θ is continuous as the $A(H_i)$ are semisimple [9, p. 76]. Let us call an isomorphism from $A(H_1)$ to $A(H_2)$ induced from θ by $f \to f \circ \theta$ a *standard isomorphism*. The existence of a standard isomorphism taking $A(H_2)$ into $A(H_1)$ is therefore equivalent to $\Theta(A(H_2)) \subseteq A(H_1)$.

If Θ is an isomorphism from $A(H_2) \to A(H_1)$ then Θ^* must be a bounded linear function from $N_1(H_1) \to N_1(H_2)$. The following central lemma utilizes this fact.

LEMMA 2.1. *Let θ: $H_1 \to H_2$ be a standard homeomorphism. Let Θ be the induced map from $C(H_2) \to C(H_1)$*

$$\Theta(f) = f \circ \theta, \quad f \in C(H_2).$$

Suppose $\Theta(A(H_2)) \subseteq A(H_1)$. *Let* $\{l(j)\}$, $\{n(j)\}$ *be sequences of integers with* $l(j) \leqq n(j) < l(j+1)$ *and for all* j, $n(j) - l(j) < B$ *for some constant* B. *Then, given any* $\varepsilon > 0$ *there does not exist a sequence of* $\mu_j \in M(H_1(l(j), n(j)))$ *with*

$$\| \Theta^* \mu_j \|_{PM} / \| \mu_j \|_{PM} > 1 + \varepsilon.$$

Proof.

Case 1. Assume $H_2 = E_m$ for some sequence $\{m(j)\}$. We may assume that $\| \mu_j \|_{PM} = 1$. Form the measure $\lambda_k \in N_1(H_1)$ by

$$\lambda_k = \mu_1 * \cdots * \mu_k.$$

Then

$$\| \lambda_k \|_{PM} \leqq 1.$$

Since $H_2 = E_m$, $l(j+1) > n(j)$, and Θ is a standard isomorphism, we see that

$$\| \Theta^*(\lambda_k) \|_{PM} = \prod_1^k \| \Theta^*(\mu_j) \|_{PM} > (1 + \varepsilon)^k$$

which contradicts the continuity of Θ^*.

Case 2. Assume $H_2 = F_r$. As before assume that $\| \mu_j \|_{PM} = 1$, for all j. We shall define λ_k inductively. Let λ_{k-1} be defined as a convolution of a subsequence of the μ_j by

$$\lambda_{k-1} = \mu_{j(1)} * \cdots * \mu_{j(k-1)}.$$

Assume that

$$\| \Theta^*(\lambda_{k-1}) \|_{PM} \geqq (1 + \varepsilon/8)^{k-1}$$

Choose an $\alpha > 0$ so that $(1 - \alpha)\ (1 + \varepsilon/4) > 1 + \varepsilon/8$. Since $\widehat{\Theta^*(\lambda_{k-1})}(x)$ is an almost periodic function of x there is an N so that for any real y

$$\sup_{|x-y| \leqq N} | \widehat{\Theta^*(\lambda_{k-1})}(x) | > (1 - \alpha) \| \Theta^*(\lambda_{k-1}) \|_{PM}.$$

Pick $j(k)$ so that

$$N \cdot r(l(j(k))) < \varepsilon / 16\pi 2^B \cdot (B + 1). \tag{2.2}$$

Pick a y so that $| \widehat{\Theta^*(\mu_{j(k)})}(y) | > 1 + \varepsilon/2$. There is some x with $|x - y| \leqq N$ for which

$$| \widehat{\Theta^*(\lambda_{k-1})}(x) | \geqq (1 - \alpha)(1 + \varepsilon/8)^{k-1}.$$

However

$$\left|\widehat{\Theta^*(\mu_{j(k)})}(x) - \widehat{\Theta^*(\mu_{j(k)})}(y)\right| \leqq 2^B \cdot 4\pi \cdot (B+1) \cdot N \cdot r(l(j(k))$$

which by (2.2) is less than or equal to $\varepsilon/4$. Set

$$\lambda_k = \lambda_{k-1} * \mu_{j(k)}.$$

Then

$$\begin{aligned}\left|\widehat{\Theta^*(\lambda_k)}(x)\right| &= \left|\widehat{\Theta^*(\lambda_{k-1})}(x)\right| \cdot \left|\widehat{\Theta^*(\mu_{j(k)})}(x)\right| \\ &\geqq (1+\varepsilon/8)^{k-1}(1-\alpha)(1+\varepsilon/4) \\ &\geqq (1+\varepsilon/8)^k.\end{aligned}$$

However

$$\begin{aligned}\|\lambda_k\|_{PM} &\leqq \prod_{s=1}^{k} \|\mu_{j(s)}\|_{PM} \\ &\leqq 1.\end{aligned}$$

Therefore, $\|\Theta^*(\lambda_k)\|_{PM}/\|\lambda_k\|_{PM} \to \infty$ as $k \to \infty$ and the continuity of Θ^* is contradicted. Q.E.D.

The next lemma is found in [5] but we include a proof for the sake of completeness. We let E_1, E_2 be compact subsets of locally compact groups G_1, G_2. Let ϕ: $E_2 \to E_1$ be a homeomorphism. Assume that $0 \in E_2$, $0 \in E_1$ and $\phi(0) = 0$. Let Φ be the map from $A(E_1) \to C(E_2)$ with $f \to f \circ \phi$. If $\gamma \in \Gamma_1$ we note that $\gamma^1 = \gamma/E_1$ is an element of $A(E_1)$ and define $\Phi(\gamma)$ by $\Phi(\gamma^1)$.

LEMMA 2.3. *If $\Phi: A(E_1) \to A(E_2)$ is an isomorphism into of norm 1, then for every $\gamma \in \Gamma_1$, $\Phi(\gamma)$ is the restriction to E_2 of an element of the Bohr compactification of Γ_2.*

Proof. Since E_1 is compact we see by [15, p. 53] that $\|\gamma\|_{A(E_1)} = 1$. Therefore $\|\Phi(\gamma)\|_{A(E_2)} = 1$. Let $\{\hat{f}_n\} \in A(G_2)$ be a sequence of elements with $\|\hat{f}_n\|_{A(G_2)} \leqq 1 + 1/n$ and $\hat{f}_n \mid E_2 = \Phi(\gamma)$. Consider the f_n as measures on $\tilde{\Gamma}_2$, the Bohr compactification of Γ_2. Since $\tilde{\Gamma}_2$ is compact, there is a $\mu \in M(\tilde{\Gamma}_2)$ that is a weak* limit point of $\{f_n\}$. We see that $\|\mu\|_{M(\tilde{\Gamma}_2)} \leqq 1$ and $\hat{\mu}/E_2 = \Phi(\gamma)$. However by the assumption that $\phi: 0 \to 0$ we see $\hat{\mu}(0) = 1$ and hence μ is a positive measure. From [5, p. 123] we see that there is a closed subgroup G_μ with $E_2 \subset G_\mu \subset G_2$ on which $\hat{\mu}$ is multiplicative. By [14, p. 138] $\Phi(\gamma)$ is the restriction to E_2 of an element of $\tilde{\Gamma}_2$. Q.E.D.

THEOREM 2.4. *Let $\{m(j)\}$ be a bounded sequence of integers, $\phi: E_m \to F$*

a standard homeomorphism. Then the induced map $\Phi: C(F) \to C(E)$ does not take $A(F)$ into $A(E)$.

Proof. Assume $\Phi(A(F)) \subset A(E)$.

Since $m(j)$ is bounded, we can find an infinite number of $j(i)$ with $m(j(i)) = n$. Consider the two point measures $\alpha_i \in N_1(E)$,

$$\alpha_i\{0\} = \frac{1}{2}$$

$$\alpha_i\{x^{j(i)}\} = \frac{1}{2}\exp(1/3n)$$

where for any integer s, $x_j^s = \delta_j^s$. It is easy to see that

$$\| \alpha_i \|_{PM} \leqq 1 - 1/C,$$

where C is a constant only depending on n. However

$$\| \Phi^*(\alpha_i) \|_{PM} = 1$$

which contradicts Lemma 2.1. Q.E.D.

If γ is a constant with $\gamma < \frac{1}{2}$ and $\{r(j);\ j = 1, \cdots\}$ is the sequence defined by $r(j) = \gamma^j$, F_r is called a *symmetric set of constant ratio.* Our next theorem concerns standard isomorphisms between two restriction algebras of symmetric sets of constant ratio.

THEOREM 2.5. *Suppose that $\gamma < \frac{1}{2}$, $\lambda < \frac{1}{2}$, $r(j) = \gamma^j$, and $s(j) = \lambda^j$, $\phi: F_s \to F_r$ the standard homeomorphism. Then if the induced map $\Phi: C(F_r) \to C(F_s)$ takes $A(F_r)$ into $A(F_s)$, γ satisfies the irreducible equation of λ over the rationals.*

Proof. If λ is transcendental the theorem is vacuously true. Otherwise, let $\sum_0^{k-1} C_j x^j$ with C_j integers, be the irreducible equation of λ.

Φ induces a map Φ' from $A(F_r^k) \to A(F_s^k)$. If the norm of Φ' is greater than one, there would be a $\nu \in M(F_s^k)$ with

$$\| \Phi^*(\nu) \|_{PM} / \| \nu \|_{PM} > 1 + \varepsilon.$$

Set

$$\nu_j(\{\lambda^{jk} x\}) = \nu(\{x\}).$$

We see that

$$\| \nu_j \|_{PM} = \| \nu \|_{PM}$$

and

$$\| \Phi^*(\nu_j) \|_{PM} = \| \Phi^*(\nu) \|_{PM}$$

which contradicts Lemma 2.1. Therefore Φ' is of norm one. Let ϕ be the standard homeomorphism taking $F_s \to F_r$ and ϕ' its restriction to F_s^k. Lemma 2.3 shows that there is for each $x \in R$ a (not necessarily continuous) character ψ_x of R for which

$$\psi_x(\lambda^j) = \exp(x\gamma^j); \qquad j = 1, \cdots, k.$$

Since $\sum_1^k C_{j-1} \lambda^j = 0$

$$\psi_x \left(\sum_1^k C_{j-1} \lambda^j \right) = 1$$

$$\prod_1^k \psi_x(\lambda^j)^{C_{j-1}} = 1$$

$$\exp\left(x \sum_1^k C_{j-1} \gamma^j \right) = 1$$

Since x is arbitrary

$$\sum_1^k C_{j-1} \gamma^j = 0$$

and hence γ satisfies the irreducible equation of λ. Q.E.D.

It is easy to see by Kronecker's Theorem [4, p. 99] that if γ and λ are conjugates then the map ϕ: $N_1(F_s) \to N_1(F_r)$ and the map ϕ^{-1} are bounded. We have not been able to obtain isomorphisms of $A(F_r)$ and $A(F_s)$ when γ and λ are conjugates.

THEOREM 2.6. *Let $\gamma < \frac{1}{2}$, $r(j) = \gamma^{-j}$; $j = 1, 2, \cdots$. Let $m = m(j)$; $j = 1, 2, \cdots$ be any sequence of positive integers, $\psi: F_r \to E_m$ the standard homeomorphism. Then if the induced map Ψ: $C(E_m) \to C(F_r)$ takes $A(E_m)$ into $A(F_r)$, γ must be transcendental.*

Proof. Suppose $\Psi(A(E_m)) \subset A(F_r)$ and γ is algebraic. Let $\sum_1^k c_j \gamma^j = 0$ where the c_j are relatively prime integers.

By diagnolizing one can find a sequence $E_m(l(j), l(j) + 3k)$ with $l(j+1) > l(j) + 3k$ and for each $r = 0, \cdots, 3k$ either

$$m(l(j) + r) = m(l(i) + r) \tag{2.7}$$

for all i,j or

$$\lim_{j \to \infty} m(l(j) + r) = \infty. \tag{2.8}$$

Ψ induces the standard isomorphism

$$\Psi' \colon A(E(l(1), l(1) + 3k)) \to A(F(l(1), l(1) + 3k)).$$

We claim that $\| \Psi' \|$ is one. For ease of notation, denote $E(l(j), l(j) + 3k)$ by $E(j)$ and similarly with $F(j)$. If $\| \Psi' \| > 1$ there exists $v \in M(F(1))$ with $\| v \|_{PM} = 1$ and

$$\| \Psi^*(v) \|_{PM} > 1 + \varepsilon.$$

Let $v_j \in M(F(j))$ be defined by

$$v_j(\gamma^{(l(j)-l(1))} \cdot x) = v(x).$$

It is clear that

$$\| v_j \|_{PM} = \| v \|_{PM} = 1.$$

We see from (2.7), and (2.8) and continuity considerations that for j large enough

$$\| \Psi^*(v_j) \|_{PM} > 1 + \varepsilon/2.$$

Applying Lemma 2.1 we arrive at a contradiction and hence $\| \Psi' \| = 1$. Let α be the character of $\prod Z_{m(j)}$ with

$$\alpha_j = \begin{cases} 0 & \text{if } \ j \neq l(1) + k \\ 1 & \text{if } \ j = l(1) + k. \end{cases}$$

By Lemma 2.2, $\Psi'(\alpha_j)$ is the restriction to $F(1)$ of a character θ of the discrete real numbers. If $l(1) \leqq r \leqq l(1) + 3k$

$$\theta(\gamma^{l(1)+r}) = \begin{cases} 1 & \text{if } \ r \neq k \\ \exp(1/m(l(1)+k)) & \text{if } \ r = k. \end{cases}$$

Since $\sum_1^k c_j \gamma^j = 0$, then $\sum_1^k c_j \gamma^{l(1)+r+j} = 0$ for any r. If $0 < r \leqq k - 1$

$$\begin{aligned} 1 &= \theta\left(\sum_1^k c_j \gamma^{l(1)+r+j} \right) \\ &= \prod_1^k (\theta(\gamma^{l(1)+r+j}))^{c_j} \\ &= \exp(c_{k-r}/m(l(1)+k)). \end{aligned}$$

Therefore $m(l(1) + k)$ divides c_{k-r} for each $0 \leqq r \leqq k - 1$. Since the c_j were assumed relatively prime and $m(j) \geqq 2$, there is a contradiction. Q.E.D.

If the γ of Theorem 2.6 is transcendental, then $\psi: N_1(F) \to N_1(E)$ is of norm one. We do not know if this extends to an isomorphism of $A(E) \to A(F)$.

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